would be acceptable for occultation to occur. During this interval, the period control could be powered down (i.e., not used), and fuel would be saved; however, when the satellite was to be used again, a large ΔV would be required to achieve transfer from the occulting orbit to an occultation-free orbit. Nevertheless, it is found that power-down produces a net fuel saving whenever the time of limited use exceeds the nominal interval-between-thrusts.

Relation to Literature

These results are in conflict with a period control proposed by Farquhar ("Lunar Communications with Libration Point Satellites," *Journal of Spacecraft and Rockets*, Vol. 4, No. 10, October 1967, pp. 1383–1384). Farquhar's solution, however, involves substantially higher cost and in addition represents a single optimum and not a family of local optima. Moreover, his solution is objectionable on theoretical grounds since its characteristics are strongly dependent upon the small parameter e, the lunar eccentricity, and his solution actually fails to exist for $e = 0$. The preceding solution, though computed for the case $e = 0$, requires only slight modification to accommodate the actual case of $e \ll 1$ (i.e., A_z must be increased). In this light, Farquhar's requirement that $e > 0$ appears artifical.

Optimal Controls for Out-of-Plane Motion about the Translunar Libration Point

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For a spacecraft in stabilized motion about the lunar L_2 point, the periods of the in-plane and out-of-plane oscillations must be synchronized to prevent occultation of the spacecraft by the moon. Phase-plane methods are used to construct a family of locally fuel-optimal out-of-plane period controls, implemented with impulsive thrust. A tradeoff between fuel penalty and interval-between-thrusts is derived, and it is shown that the effects of anomalous thrust vectors may be easily dealt with. For operational reasons, impulses may have to be made at nonoptimum times; it is shown that the associated fuel penalty can be bounded above. During times of limited use of the satellite it may be possible to power-down the period control; it is shown how power-up may be accomplished so that there is a net saving in fuel. Results are plotted in a form useful for mission planning and operation.

Introduction

A TOPIC of some current interest is the establishment and control of a satellite in the vicinity of the translunar libration point or L_2 point, which is one of the five points of equilibrium in the Earth-moon gravitational field. Such a satellite could be used for radio astronomy, communications with the lunar farside, solar-wind studies, and the like. Among the most detailed studies of the astrodynamics problems associated with such satellites have been those of Farquhar[1] and of General Electric.[2]

To the accuracy required by a preliminary study, motion in the vicinity of the L_2 point may be studied by means of a restricted three-body problem model wherein the effects of lunar eccentricity ($e = 0.05490$) are implicitly neglected, as are the perturbing effects of fourth bodies (e.g., the Sun). Near the L_2 point the governing equations of motion may be linearized. Following the derivation given by Szebehely,[3] these linearized equations of motion take the form

$$\ddot{x} - 2\dot{y} - 7.38085x = 0 \qquad (1)$$

$$\ddot{y} + 2\dot{x} + 2.19042y = 0 \qquad (2)$$

$$\ddot{z} + 3.19042z = 0 \qquad (3)$$

where x,y,z form a right-handed system as in Fig. 1, and time t is normalized with respect to the lunar mean motion; unit time is 4.34838 days.

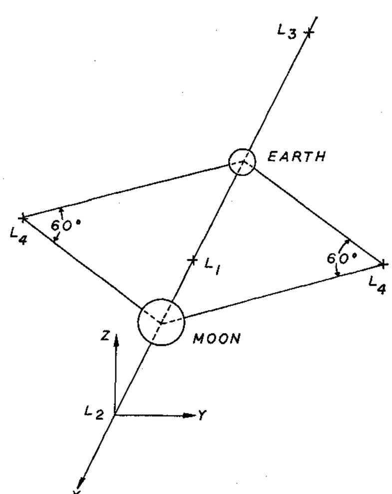

Fig. 1 The five libration points in the Earth-moon system, and the coordinate system used in this paper.

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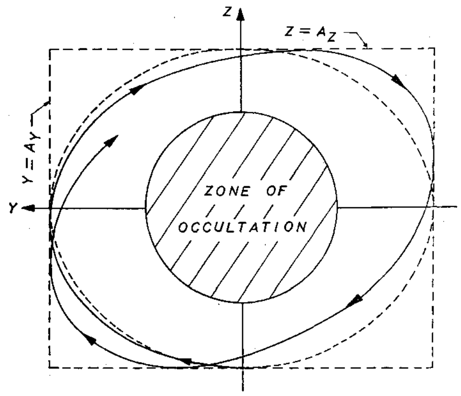

Fig. 2 In the absence of period control, motion in the y-z plane follows a Lissajous curve bounded by the dotted rectangle.

Equations (1) and (2) are coupled and define a linearly unstable motion. However, there exists a one-parameter family of periodic orbits in the x-y plane, with frequency $\omega_y = 1.86265$, which are produced under suitable initial conditions. Numerical studies[4] have shown, however, that neglected perturbations will destabilize the motion, imparting a strongly divergent and nonperiodic character after the satellite has completed no more than approximately one orbit. Therefore it is necessary to employ a stationkeeping control so as to stabilize the x-y motion about a reference orbit chosen from this family, and the cited references contain a number of proposals for thrust-implemented stationkeeping systems.

Equation (3) is uncoupled and represents simple harmonic, linearly neutrally stable motion with frequency $\omega_z = 1.78618$. For a satellite executing simultaneous y and z oscillations, the motion normal to the Earth-moon line of centers describes a Lissajous curve (Fig. 2), and the satellite will therefore be occulted on occasion. For many applications this condition is undesirable; it is therefore necessary to implement a period control so as to synchronize the y and z oscillations. A variety of period controls have been proposed in the literature,[1,2,5,6] for which the fuel budget is one to two orders of magnitude greater than that of the stationkeeping system. In view of the quite different functions and fuel budgets for period control and for stationkeeping, it is desirable to decouple their effects upon the motion, and this can be done by use of a z-period control. A z-period control is also advantageous in that it can provide operational flexibility by adjusting the z-amplitude A_z, and in this paper a family of locally fuel-optimal z-period controls is constructed and their properties investigated.

It should be noted that the primary motivation for such controls is the necessity of preventing occultation of a lunar farside communications link, such as was first proposed by Farquhar.[6] Such a link clearly cannot be used during periods of occultation; additionally, occultation prevents continuous tracking and thus diminishes tracking accuracy, with consequent degradation of the stationkeeping fuel economy.[1] Farquhar[9] has proposed additional applications for libration point satellites, for which occultation also may be undesirable.

Optimal Controls for the z Motion

A theorem due to Neustadt[7] states that if, in a linearized orbit-transfer problem, k final states are prescribed, the optimal number of impulses required for the transfer is $\leq k$. For this problem $k = 2$ since in general the z-period control will be used for transfer to given final values of z, $\dot{z}$.

From Eq. (3), $z = z(t)$ and $\dot{z} = \dot{z}(t)$ are given as

$$z = A_z \cos(\omega_z t + \alpha_0) \qquad \dot{z} = -\omega_z A_z \sin(\omega_z t + \alpha_0) \quad (4)$$

where α_0 is a phase angle. Defining $\theta \equiv \omega_z t + \alpha_0$ and eliminating θ gives the result, $(z/A_z)^2 + (\dot{z}/\omega_z A_z)^2 = 1$, which defines the phase trajectories as a family of concentric circles in $(z, \dot{z}/\omega_z)$ phase space with parameter A_z. Figure 3a shows a general case wherein two impulses change the amplitude from A_{z1} to A_{z3} while reducing the period of one orbit by $(\Delta_a + \Delta_b)$. In such a general case, the ΔV for the two impulses is given as

$$\Delta V/\omega_z = |A_{z1} \sin\theta_1 - A_{z2} \sin(\theta_1 + \omega_z \Delta_a)| + |A_{z2} \sin\theta_2 - A_{z3} \sin(\theta_2 + \omega_z \Delta_b)| \quad (5)$$

and in what follows, in discussing individual thrusts, A_{z1} and A_{z2} will be understood to represent A_z, respectively, before and after the thrust. For a periodic orbit the condition $A_{z3} = A_{z1}$ is imposed.

The period-control problem consists of reducing the z period:

$$\Delta(z \text{ period}) \equiv \Delta_n = 2n\pi[(1/\omega_z) - (1/\omega_y)] \quad n = 1,2,\ldots \quad (6)$$

so that the y and z oscillations are synchronized over the time of n orbits. As mentioned previously, $A_{z3} = A_{z1}$; moreover, $A_{z1}, A_{z2} \geq A_{z0}$ where A_{z0} guarantees nonoccultation. (The selection of a suitable A_{z0} is discussed below.) An additional constraint is that $\Delta_a + \Delta_b = \Delta_n$, where Δ_n is given by Eq.

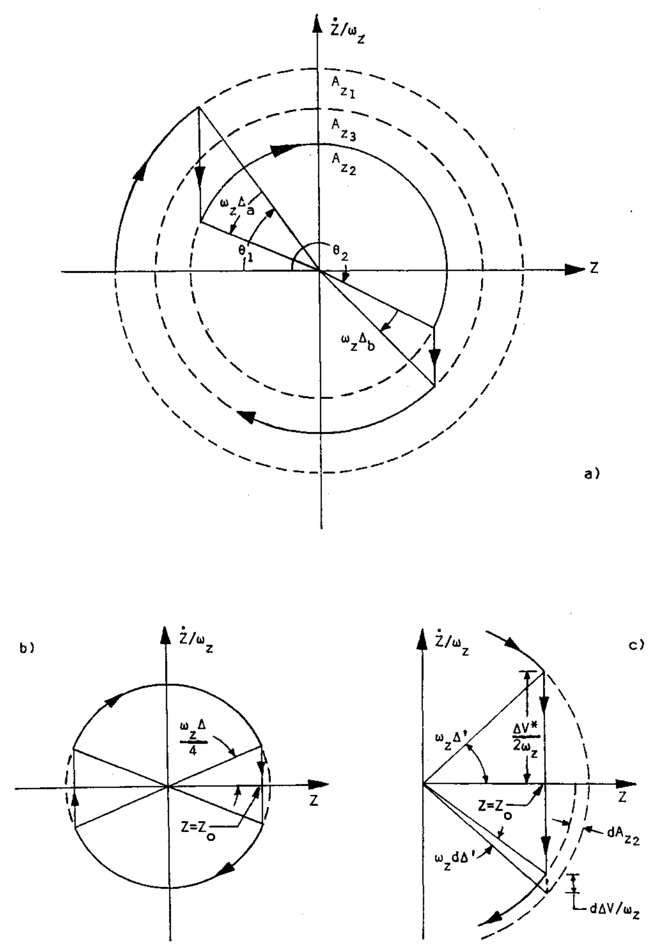

Fig. 3 Motion in the (z, $\dot{z}/\omega_z$) phase plane. a) Genera case of the use of two impulses to change the amplitude from A_{z1} to A_{z3} while reducing the period for one orbit by ($\Delta_a + \Delta_b$). b) Optimal reduction in period by Δ, under the constraints $A_{z1} = A_{z3}$ and A_{z1}, $A_{z2} \geq A_{z0}$. c) Effect on A_{z2} of an anomaly in ΔV.

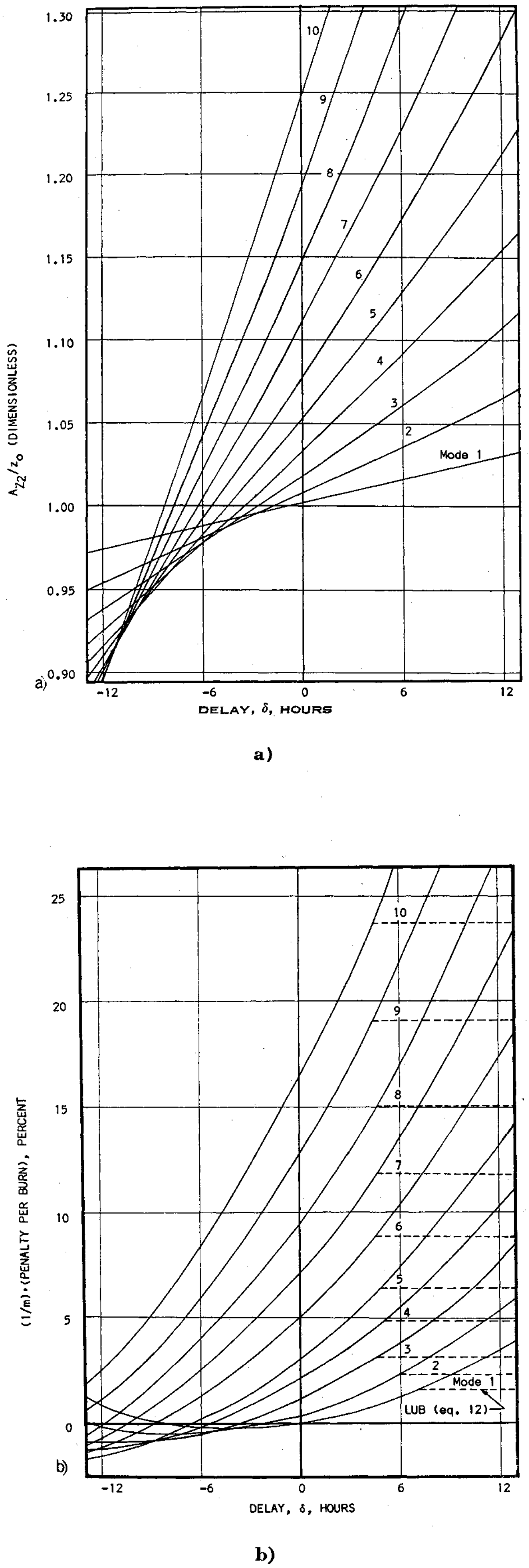

Fig. 4 Effect on fuel economy and amplitude A_{z2} of period-control thrusts made at nonoptimal times. a) Post-burn amplitude A_{z2}; z_0 = const. b) Fuel penalty with respect to the globally optimal Mode 1.

(6). The construction of the optimal period control is thus reduced to an algebraic minimization problem with inequality constraints, and the solution is found to be

$$\Delta_a = \Delta_b = \Delta_n/2, \quad \theta_1 = -\omega_z\Delta_n/4, \quad \theta_2 = \pi - \omega_z\Delta_n/4, \quad A_{z1} = A_{z2} = A_{z0} \quad (7)$$

(Fig. 3b). The assumption that the optimum exists for either or both of $A_{z1}, A_{z2} > A_{z0}$ will in fact lead to a contradiction of that assumption, or a violation of the constraint $\Delta_a + \Delta_b = \Delta_n$.

It should be noted that less restrictive constraints lead to different optima. If $A_{z3} = A_{z1} \geq A_{z0}$ but A_{z2} is unconstrained, and $\Delta_a + \Delta_b = \Delta_n$, the optimal solution is given by $\Delta_a = \Delta_b = \Delta_n/2$, $\theta_1 = -\omega_z\Delta_n/2$, $\theta_2 = \pi$, $A_{z1} = A_{z0}$, $A_{z2} = A_{z0}\cos(\omega_z\Delta_n/2)$. For a change in amplitude from A_{z1} to A_{z3}, both values being given, and again with $\Delta_a + \Delta_b = \Delta_n$, the optimum transfer involves a single impulse with θ_1 given by the expression $A_{z1}\cos\theta_1 = A_{z3}\cos(\theta_1 + \omega_z\Delta_n)$; no fuel savings can be attained by a two-impulse transfer involving an intermediate amplitude A_{z2}. Finally, if the constraint $\Delta_a + \Delta_b = \Delta_n$ is removed, the optimal amplitude-change maneuver is a single impulse applied at $\theta_1 = \pm\pi/2$, with $\Delta_a = \Delta_b = 0$.

Equations (5) and (7) give the optimal cost for n orbits:

$$\Delta V = 4\omega_z z_0 \tan(\omega_z\Delta_n/4) \quad n = 1,2,\ldots \quad (8)$$

where $z_0 = A_{z0}\cos(\omega_z\Delta_n/4)$ is the maximum value of z attained during an orbit (Fig. 3b). In what follows, $z_0 = 4000$ km will be taken. The radius of the occulted zone, within which the moon wholly or partially eclipses the earth, is 3100 km at L_2 distance; the additional 900 km serves as a buffer zone which ensures that neglected perturbations will not produce occultation. See Farquhar,[1] p. 178, for further discussion on this point.

Equations (6) and (8) define a family of locally fuel-optimal period controls in the parameter n. The jth member of this family will be referred to as Mode j; it involves reversals in $\dot{z}$ made every $j/2$ orbits; or, since the average orbital period must be $2\pi/\omega_y = 14.6683$ days, $\dot{z}$ is reversed every $7.3341j$ days. It is clear that Mode 1 is globally optimal, with a cost given by $\Delta V = (100.2z_0)$ fps/year, with z_0 in units of 10^3 km.

From considerations of engine reliability and navigational accuracy, it may be advantageous to thrust as infrequently as possible. Equations (6) and (8) serve to define a tradeoff between fuel penalty and interval between thrusts, the fuel penalty being defined with respect to Mode 1.

$$\text{Mode } n \text{ fuel penalty} = [\tan(n\omega_z\Delta_1/4)/n\tan(\omega_z\Delta_1/4)] - 1 \quad (9)$$

where $\omega_z\Delta_1 = 0.25796$. Equation (9) defines the points lying on the ordinate of Fig. 4b; thus it is seen, for example, that the interval between thrusts may be increased tenfold for a penalty of 16.47%. In what follows it will be assumed that Mode m has been selected as nominal on the basis of a trade study, where m = const.

Non-Nominal Implementation of the Control

Three important ways in which a period control thrust may be off-nominal are that it may produce an anomalous ΔV, its direction may be anomalous, or it may be made at a nononominal time. This last anomaly is especially important for the case of a satellite with a large fixed parabolic antenna which is utilized for extended periods; for such a satellite, the attitude changes associated with a thrust would interfere with use of the antenna, so that it could be necessary to advance or delay the thrust.

Anomalous ΔV Magnitude

Let $\Delta V = \Delta V^*$ be nominal, and let the actual value of ΔV have a negative anomaly, $\Delta V = \Delta V^* - d\Delta V$. To determine its effect on A_{z_2}, consider Fig. 3c. From this figure,

$$dA/d\Delta V = (1/\omega_z)\sin(\omega_z\Delta')$$

For the somewhat extreme case of Mode 10, $\Delta' = 1.57$ days, and if (for example) $d\Delta V = 0.5$ fps, then $dA_{z_3} = 62{,}900$ ft. This is of the same order of magnitude as the neglected solar perturbation,[8] so it is concluded that the effects of anomalous ΔV may fall into the category of negligible perturbations. If the effect is considered nonnegligible, it can be nulled out by introducing an equal positive anomaly into the next scheduled thrust. It should be noted that if the probable magnitude error $(d\Delta V/\Delta V) =$ const. for all ΔV, then $dA_{z_2} \propto n^2$ for Mode n, approximately.

Anomalous Thrust Directions

These will produce a component in the x-y plane of $(\Delta V)_{x-y} = \Delta V d\psi$, where $d\psi$ = error in thrust direction, assumed small. For $d\psi = 1°$ and $\Delta V = 401$ fps/yr (i.e., $z_0 = 4000$ km), $(\Delta V)_{x-y} = 7.0$ fps/yr. Typical values of the stationkeeping budget have been given as 25 fps/yr (Ref. 6) and 60 fps/yr,[2] so it is seen that the effects of anomalies in the thrust direction fall into the category of velocity perturbations which constitute inputs to the stationkeeping system.

Thrusts at Non-Nominal Times

Let a scheduled thrust in Mode m be delayed a time δ past the nominal. From Fig. 3, with $A_{z_1} = A_{z_0}$, the following relations exist:

$$A_{z_3}\cos\omega_z[(m\Delta_1/4) - \delta] = A_{z_2}\cos\omega_z[(m\Delta_1/4) + \delta] \quad (10)$$

$$\Delta\dot{z}/\omega_z = A_{z_0}\sin\omega_z[(m\Delta_1/4) - \delta] + A_{z_2}\sin\omega_z[(m\Delta_1/4) + \delta] \quad (11)$$

Let A_{z_0}, A_{z_2} be normalized with respect to z_0, and $\dot{z}$ normalized with respect to $(\Delta\dot{z})_m^* = 2m\omega_z z_0 \tan(\omega_z\Delta_1/4)$, the cost of Mode 1 for $m/2$ orbits. It is seen that $\Delta\dot{z}/(\Delta\dot{z})_m^*$ and A_{z_2}/z_0 are monotonically increasing functions of δ for $\delta \geq 0$; these functions are plotted in Fig. 4 for the first 10 modes, where $[\Delta\dot{z}/(\Delta\dot{z})_m^* - 1]$ is the ordinate of Fig. 4b; $(\Delta\dot{z})_1^*$ is unity. If this change in amplitude is not desired, it may be nulled out with another nonoptimal burn made $m/2$ orbits later and, in the phase plane, being symmetric about the z-axis with respect to the first burn. In what follows it is assumed that this is done.

The associated fuel penalty can be bounded above if, for sufficiently large δ, the scheduled burn is cancelled and the next two burns are made in Mode $(m + 1)$. To ensure non-occultation, these burns must be made at $z \geq z_0$; the least upper bound then occurs at $z = z_0$, and is given by

$$LUB = \frac{\mathrm{m}+1}{\mathrm{m}}\left\{\frac{\omega_z z_0}{(\Delta\dot{z})^*_{m+1}}\left[\tan\frac{m\omega_z\Delta_1}{4} + \tan\frac{(m+2)\omega_z\Delta_1}{4}\right] - 1\right\} \quad (12)$$

Here $LUB = 1/2m$ times the total penalty associated with the delay. Equation (12) defines the dashed horizontal lines in Fig. 4b.

An advance in the scheduled Mode m burn (i.e., $\delta < 0$) will save fuel but reduce A_{z_2}; thus, sufficiently large advances will produce occultation. A lower bound for δ must be computed for specific cases through solution of the exact equations of motion, since neglected perturbations (chiefly the lunar eccentricity perturbation) will strongly influence the existence and length of any period of occultation resulting from taking $\delta < 0$.

Occasionally Occulted Orbits

It is possible that there will be long intervals when use of the satellite is so limited that occultation is permissible, e.g., if the satellite is used chiefly to support Apollo missions. During such intervals the period-control system may be powered down, but when the satellite is to go on-line again it will be necessary to transfer out of the occulting orbit, a maneuver which in general requires a large ΔV. It is of interest whether a saving in fuel may be achieved by means of such a power-down.

This problem is studied in the physical plane (Fig. 2). From Eqs. (2) and (4), motion in the y-z plane is defined by

$$y = A_y \sin\omega_y(t + t_0) \quad (13)$$

$$z = A_z \cos[\omega_y(t + t_0) + \alpha] \quad (14)$$

where $\alpha \equiv \alpha_0 + (\omega_z - \omega_y)(t + t_0)$. At the time of transfer out of the occulting orbit (power-up), motion is initiated with $\alpha = 0$; this condition defines the dotted ellipse in Fig. 2 as the locus of points where power-up may take place into orbits with amplitudes A_z, A_y. On this ellipse, $[y(t)/A_y]^2 + [z(t)/A_z]^2 = 1$; applying this condition to Eqs. (13) and (14) defines the times of possible power-ups

$$\cos^2\omega_y(t + t_0) = \cos^2[\omega_z(t + t_0) + \alpha_0] \quad (15)$$

Equation (15) has the solution

$$t_k + t_0 = (k\pi - \alpha_0)/(\omega_z + \omega_y),\ k = 1,2,\ldots \quad (16)$$

so that, for this problem, power-ups are possible every 3.745 days following the initial intersection with the dotted ellipse, t_k defining the time of the kth intersection. At power-up the velocity $\dot{z}$ is found from Eqs. (14) and (16)

$$\dot{z}_P = \omega_z A_z \sin[\omega_z(t_k + t_0) + \alpha_0] \quad (17)$$

whereas the desired velocity, corresponding to $\alpha = 0$, is given by

$$\dot{z}_D = \pm\omega_z A_z \sin\omega_y(t_k + t_0) \quad (18)$$

where the ambiguous sign derives from Eq. (15) and physically signifies that although the initial motion was clockwise, power-up into both clockwise and counterclockwise orbits is possible. The ΔV requirement for power-up is then $\Delta V = \dot{z}_D - \dot{z}_P$; algebraic manipulations give the result

$$\begin{aligned} \Delta V &= 2\omega_z A_z \sin\tfrac{1}{2}[(\omega_z - \omega_y)(t_k + t_0) + \alpha_0] \quad & k = 0,2,4,\ldots \\ \Delta V &= 2\omega_z A_z \cos\tfrac{1}{2}[(\omega_z - \omega_y)(t_k + t_0) + \alpha_0] \quad & k = 1,3,5,\ldots \end{aligned} \quad (19)$$

Thus, clockwise power-ups are possible for $k = 0,2,4,\ldots$, counterclockwise power-ups are possible for $k = 1,3,5,\ldots$, and at no time are both types possible with a $\dot{z}$ control. For this problem, $(\omega_y - \omega_z)/2 = (113.8 \text{ days})^{-1}$. Let $\gamma \equiv \frac{1}{2}[(\omega_z - \omega_y)(t_n + t_0) + \alpha_0]$; optimum power-ups are then clockwise for $|\gamma| < \pi/4$ and for $3\pi/4 < |\gamma| < 5\pi/4$, counterclockwise at other values of γ (assuming an initially clockwise motion). It may be seen that power-ups made in this manner are such that there is a net saving in fuel for any power-down interval greater than the nominal interval-between-thrusts for Mode m. Similar results have been found by Opdycke.[6]

Conclusions

It is seen that use of a z-period control provides an effective and convenient method for prevention of occultation. Considerable operational flexibility is provided, since any single burn may be used to adjust period, amplitude, or both. Increased engine reliability, which may result from a decreased cycling rate, can be attained with modest fuel penalties. An actual satellite may not require an articulable high-gain antenna even though such an antenna would permit the period control thrusts to be made always at optimum times, because, as Fig. 4b demonstrates, the added penalty per burn associated with even very long delays may be given (very roughly) for Mode m by $(m/2)\%$. Increases in the nominal amplitude A_{z0} will increase the interval prior to the nominal thrust time wherein thrusts can be made without producing occultation; such increases thus provide additional mission flexibility. The effects of anomalous thrust vectors either constitute a negligible perturbation or else can readily be nulled out. Finally, during times of limited use wherein occultation is permissible, the period-control system may be powered down because power-up may always be accomplished in such a way as to produce a net saving in fuel.

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